

On quantum channels reversible with respect to a given family of pure states.

M.E. Shirokov*

Steklov Mathematical Institute, RAS, Moscow

Abstract

A description of all quantum channels reversible with respect to a given complete family of pure states is obtained. Some applications in quantum information theory are considered.

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1 Introduction

A reversibility (sufficiency) of a quantum channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ with respect to a family \mathfrak{S} of states in $\mathfrak{S}(\mathcal{H}_A)$ means existence of a quantum channel $\Psi : \mathfrak{S}(\mathcal{H}_B) \rightarrow \mathfrak{S}(\mathcal{H}_A)$ such that $\Psi(\Phi(\rho)) = \rho$ for all $\rho \in \mathfrak{S}$.

*email:msh@mi.ras.ru

The notion of reversibility of a channel naturally arises in analysis of different general questions of quantum information theory and quantum statistics [2, 8, 9, 10, 11]. For example, the famous Petz's theorem states that the equality in the inequality

$$H(\Phi(\rho) \parallel \Phi(\sigma)) \leq H(\rho \parallel \sigma), \quad \rho, \sigma \in \mathfrak{S}(\mathcal{H}_A),$$

expressing the fundamental monotonicity property of the relative entropy, holds if and only if the channel Φ is reversible with respect to the states ρ and σ . It follows that the Holevo quantity of an ensemble of quantum states (providing an upper bound for accessible classical information which can be obtained by applying a quantum measurement [3]) is preserved under action of a channel Φ if and only if the channel Φ is reversible with respect to all states of this ensemble. Further analysis shows that preserving conditions for many others important characteristics under action of a quantum channel are also reduced to the reversibility condition [2, 9]. In [12] it is shown that necessary and sufficient conditions for coincidence of the Holevo capacity and the entanglement-assisted classical capacity of a quantum channel Φ can be formulated in terms of reversibility of the complementary channel $\hat{\Phi}$ with respect to particular families of pure states.

A simple necessary condition for reversibility of a quantum channel with respect to a family of states with bounded rank obtained by using Petz's theorem is presented in [13, Theorem 1]. In this paper we use this condition and some other observations to obtain a complete description of all channels reversible with respect to a given arbitrary family of pure states.

Some applications of the obtained results in quantum information theory are considered in the last part of the paper.

2 Preliminaries

Let \mathcal{H} be either a finite dimensional or separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ – the Banach spaces of all bounded operators in \mathcal{H} and of all trace-class operators in \mathcal{H} correspondingly, $\mathfrak{S}(\mathcal{H})$ – the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of positive operators with unit trace called *states* [4, 7].

Denote by $I_{\mathcal{H}}$ and $\text{Id}_{\mathcal{H}}$ the unit operator in a Hilbert space \mathcal{H} and the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$ correspondingly.

A linear completely positive trace preserving map $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called *quantum channel* [4, 7].

For a given channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ the Stinespring theorem implies existence of a Hilbert space \mathcal{H}_E and of an isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(A) = \text{Tr}_{\mathcal{H}_E} V A V^*, \quad A \in \mathfrak{T}(\mathcal{H}_A). \quad (1)$$

A quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni A \mapsto \widehat{\Phi}(A) = \text{Tr}_{\mathcal{H}_B} V A V^* \in \mathfrak{T}(\mathcal{H}_E) \quad (2)$$

is called *complementary* to the channel Φ [5].¹ The complementary channel is defined uniquely in the following sense: if $\widehat{\Phi}' : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_{E'})$ is a channel defined by (2) via the Stinespring isometry $V' : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_{E'}$ then the channels $\widehat{\Phi}$ and $\widehat{\Phi}'$ are isometrically equivalent in the sense of the following definition [5].

Definition 1. Channels $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ and $\Phi' : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_{B'})$ are *isometrically equivalent* if there exists a partial isometry $W : \mathcal{H}_B \rightarrow \mathcal{H}_{B'}$ such that

$$\Phi'(A) = W \Phi(A) W^*, \quad \Phi(A) = W^* \Phi'(A) W, \quad A \in \mathfrak{T}(\mathcal{H}_A).$$

The notion of isometrical equivalence is very close to the notion of unitary equivalence. Indeed, the isometrical equivalence of the channels Φ and Φ' means unitary equivalence of these channels with the output spaces \mathcal{H}_B and $\mathcal{H}_{B'}$ replaced by their subspaces $\mathcal{H}_B^\Phi = \bigvee_{\rho \in \mathfrak{S}(\mathcal{H}_A)} \text{supp} \Phi(\rho)$ and $\mathcal{H}_{B'}^{\Phi'} = \bigvee_{\rho \in \mathfrak{S}(\mathcal{H}_A)} \text{supp} \Phi'(\rho)$.² We use the notion of isometrical equivalence, since dealing with a given representation of a quantum channel Φ it is not easy in general to determine the corresponding subspace \mathcal{H}_B^Φ .

The Stinespring representation (1) generates the Kraus representation

$$\Phi(A) = \sum_k V_k A V_k^*, \quad A \in \mathfrak{T}(\mathcal{H}), \quad (3)$$

where $\{V_k\}$ is the set of bounded linear operators from \mathcal{H}_A into \mathcal{H}_B such that $\sum_k V_k^* V_k = I_{\mathcal{H}_A}$ defined by the relation

$$\langle \varphi | V_k \psi \rangle = \langle \varphi \otimes k | V \psi \rangle, \quad \varphi \in \mathcal{H}_B, \psi \in \mathcal{H}_A,$$

¹The quantum channel $\widehat{\Phi}$ is also called *conjugate* to the channel Φ [6].

²We denote by $\text{supp} \rho$ the support of a state ρ (the subspace $(\ker \rho)^\perp$).

where $\{|k\rangle\}$ is a particular orthonormal basis in the space \mathcal{H}_E . The corresponding complementary channel is expressed as follows

$$\widehat{\Phi}(A) = \sum_{k,l} \text{Tr}[V_k A V_l^*] |k\rangle\langle l|, \quad A \in \mathfrak{T}(\mathcal{H}). \quad (4)$$

The following class of quantum channels introduced by Holevo plays an essential role in this paper [4, 7].

Definition 2. A channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is called *classical-quantum* (briefly a *c-q channel*) if it has the following representation

$$\Phi(\rho) = \sum_{k=1}^{\dim \mathcal{H}_A} \langle k|\rho|k\rangle \sigma_k,$$

where $\{|k\rangle\}$ is an orthonormal basis in \mathcal{H}_A and $\{\sigma_k\}$ is a collection of states in $\mathfrak{S}(\mathcal{H}_B)$.

Following [9, 10] introduce the basic notion of this paper.

Definition 3. A channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is *reversible* with respect to a set $\mathfrak{S} \subseteq \mathfrak{S}(\mathcal{H}_A)$ if there exists a channel $\Psi : \mathfrak{S}(\mathcal{H}_B) \rightarrow \mathfrak{S}(\mathcal{H}_A)$ such that $\rho = \Psi \circ \Phi(\rho)$ for all $\rho \in \mathfrak{S}$.³

Note that the reversibility is a common property for isometrically equivalent channels.

Lemma 1. [13] *Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ and $\Phi' : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_{B'})$ be quantum channels isometrically equivalent in the sense of Def.1. If the channel Φ is reversible with respect to a set $\mathfrak{S} \subseteq \mathfrak{S}(\mathcal{H}_A)$ then the channel Φ' is reversible with respect to the set \mathfrak{S} and vice versa.*

The main result concerning the notion of reversibility is the Petz's theorem, which will be used in this paper in the following reduced form.

Theorem 1. [11] *A quantum channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is reversible with respect to full rank states ρ and σ in $\mathfrak{S}(\mathcal{H}_A)$ if and only if $\rho = \Theta_\sigma(\Phi(\rho))$, where Θ_σ is the predual channel to the linear completely positive unital map*

$$\Theta_\sigma^*(\cdot) = A\Phi(B(\cdot)B)A, \quad A = [\Phi(\sigma)]^{-1/2}, \quad B = [\sigma]^{1/2}.$$

³This property is also called sufficiency of the channel Φ with respect to the set \mathfrak{S} [8, 11].

The condition of full rank of the states ρ and σ in this theorem can be replaced by the condition $\text{supp}\rho \subseteq \text{supp}\sigma$ (see Appendix 6.1 in [13]).

Definition 4. A family \mathfrak{S} of states in $\mathfrak{S}(\mathcal{H})$ is called *complete* if for any nonzero operator A in $\mathfrak{B}_+(\mathcal{H})$ there exists a state $\rho \in \mathfrak{S}$ such that $\text{Tr}A\rho > 0$.

A family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ of pure states in $\mathfrak{S}(\mathcal{H})$ is complete if and only if the linear hull of the family $\{|\varphi_\lambda\rangle\}_{\lambda \in \Lambda}$ is dense in \mathcal{H} . By Lemma 2 in [8] an arbitrary complete family of states in $\mathfrak{S}(\mathcal{H})$ contains a countable complete subfamily.

Petz's theorem implies the following criterion for reversibility of a channel with respect to countable complete families of states.

Theorem 2. [8] *A quantum channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is reversible with respect to a complete countable family $\{\rho_i\}$ of states in $\mathfrak{S}(\mathcal{H}_A)$ if and only if $\rho_i = \Theta_{\bar{\rho}}(\Phi(\rho_i))$ for all i , where $\bar{\rho} = \sum_i \pi_i \rho_i$ and $\{\pi_i\}$ is any non-degenerate probability distribution.*

The above criterion implies the following necessary condition for reversibility of a channel with respect to families of states with bounded rank.

Theorem 3. [13] *Let $\mathfrak{S} = \{\rho_i\}_{i=1}^n$, $n \leq +\infty$, be a complete family of states in $\mathfrak{S}(\mathcal{H}_A)$ such that $\text{rank}\rho_i \leq r$ for all i . If a quantum channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is reversible with respect to \mathfrak{S} then its complementary channel $\hat{\Phi}$ has Kraus representation (3) such that $\text{rank}V_k \leq r$ for all k .*

If the above hypothesis holds with $r = 1$, i.e. $\rho_i = |\varphi_i\rangle\langle\varphi_i|$ for all i , then

$$\hat{\Phi}(\rho) = \sum_{i=1}^n \langle\phi_i|\rho|\phi_i\rangle \sum_{k=1}^{\dim \mathcal{H}_B} |\psi_{ik}\rangle\langle\psi_{ik}|,$$

where $\{|\phi_i\rangle\}_{i=1}^n$ is an overcomplete system of vectors in \mathcal{H}_A defined by means of an arbitrary non-degenerate probability distribution $\{\pi_i\}_{i=1}^n$ as follows

$$|\phi_i\rangle = \sqrt{\pi_i \bar{\rho}_\pi^{-1}} |\varphi_i\rangle, \quad \bar{\rho}_\pi = \sum_{i=1}^n \pi_i |\varphi_i\rangle\langle\varphi_i|, \quad (5)$$

and $\{|\psi_{ik}\rangle\}$ is a collection of vectors in a Hilbert space \mathcal{H}_B such that $\sum_{k=1}^{\dim \mathcal{H}_B} \|\psi_{ik}\|^2 = 1$ for all $i = \overline{1, n}$. It follows that the channel Φ is isometrically equivalent (in the sense of Def.1) to the pseudo-diagonal channel

$$\Phi'(\rho) = \sum_{i,j=1}^n \langle\phi_i|\rho|\phi_j\rangle |i\rangle\langle j| \otimes \sum_{k,l=1}^{\dim \mathcal{H}_B} \langle\psi_{jl}|\psi_{ik}\rangle |k\rangle\langle l| \quad (6)$$

from $\mathfrak{S}(\mathcal{H}_A)$ into $\mathfrak{S}(\mathcal{H}_n \otimes \mathcal{H}_B)$, where $\{|i\rangle\}_{i=1}^n$ and $\{|k\rangle\}$ are arbitrary orthonormal base in \mathcal{H}_n and in \mathcal{H}_B correspondingly.

3 Orthogonal families of pure states

A structure of a channel reversible with respect to a given complete family of orthogonal pure states is described in the following proposition.

Proposition 1. *Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel and $\mathfrak{S} = \{|\varphi_i\rangle\langle\varphi_i|\}$ be a complete family of orthogonal pure states in \mathcal{H}_A . The following statements are equivalent:*

- (i) *the channel Φ is reversible with respect to \mathfrak{S} ;*
- (ii) *$\widehat{\Phi}$ is a c-q channel having the representation $\widehat{\Phi}(\rho) = \sum_{i=1}^{\dim \mathcal{H}_A} \langle\varphi_i|\rho|\varphi_i\rangle\sigma_i$, where $\{\sigma_i\}$ is a set of states in $\mathfrak{S}(\mathcal{H}_B)$ such that $\text{rank}\sigma_i \leq \dim \mathcal{H}_B \forall i$;*
- (iii) *the channel Φ is isometrically equivalent to the pseudo-diagonal channel*

$$\Phi'(\rho) = \sum_{i,j=1}^{\dim \mathcal{H}_A} \langle\varphi_i|\rho|\varphi_j\rangle|\varphi_i\rangle\langle\varphi_j| \otimes \sum_{k,l=1}^{\dim \mathcal{H}_B} \langle\psi_{jl}|\psi_{ik}\rangle|k\rangle\langle l|$$

from $\mathfrak{S}(\mathcal{H}_A)$ into $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\{|\psi_{ik}\rangle\}$ is a collection of vectors in a separable Hilbert space such that $\sum_{k=1}^{\dim \mathcal{H}_B} \|\psi_{ik}\|^2 = 1$ for all i and $\{|k\rangle\}$ is an orthonormal basis in \mathcal{H}_B .

Proof. (i) \Rightarrow (ii) follows from Theorem 3, since in this case $\phi_i = \varphi_i \forall i$.

(ii) \Rightarrow (iii). If $\sigma_i = \sum_{k=1}^{\dim \mathcal{H}_B} |\psi_{ik}\rangle\langle\psi_{ik}|$ then $\widehat{\Phi}(\rho) = \sum_{i,k} W_{ik}\rho W_{ik}^*$, where $W_{ik} = |\psi_{ik}\rangle\langle\varphi_i|$, and hence representation (4) implies $\widehat{\widehat{\Phi}} = \Phi'$.

(iii) \Rightarrow (i) follows from Lemma 1, since $\Psi(\cdot) = \text{Tr}_{\mathcal{H}_B}(\cdot)$ is the "reverse" channel for the channel Φ' with respect to the family \mathfrak{S} . \square

Proposition 1 implies the following criterion for reversibility of a channel in terms of its dual channel.

Corollary 1. *A channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is reversible with respect to a complete family $\{|\varphi_i\rangle\langle\varphi_i|\}$ of orthogonal pure states if and only if there exists a partial isometry $W : \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}_B$ such that*

$$|\varphi_i\rangle\langle\varphi_i| = \Phi^*(W[|\varphi_i\rangle\langle\varphi_i| \otimes I_{\mathcal{H}_B}]W^*) \quad \forall i, \quad (7)$$

where $\Phi^* : \mathfrak{B}(\mathcal{H}_B) \rightarrow \mathfrak{B}(\mathcal{H}_A)$ is the dual channel to the channel Φ .

Note that condition (7) implies $\Phi^*(WW^*) = I_{\mathcal{H}_A}$ and hence WW^* is the projector on the subspace containing supports of all states $\Phi(\rho)$, $\rho \in \mathfrak{S}(\mathcal{H}_A)$.

Proof. Necessity of condition (7) directly follows from Proposition 1.

To prove its sufficiency consider the channel $\Phi'(\rho) = W^*\Phi(\rho)W$ from $\mathfrak{S}(\mathcal{H}_A)$ into $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. By the remark after Corollary 1

$$W\Phi'(\rho)W^* = WW^*\Phi(\rho)WW^* = \Phi(\rho), \quad \rho \in \mathfrak{S}(\mathcal{H}_A),$$

and hence the channels Φ and Φ' are isometrically equivalent. By Lemma 1 it suffices to show reversibility of the channel Φ' with respect to the family $\{|\varphi_i\rangle\langle\varphi_i|\}$.

Condition (7) implies

$$\text{Tr} [|\varphi_i\rangle\langle\varphi_i| \otimes I_{\mathcal{H}_B}] \Phi'(|\varphi_j\rangle\langle\varphi_j|) = \text{Tr} \Phi^*(W[|\varphi_i\rangle\langle\varphi_i| \otimes I_{\mathcal{H}_B}]W^*) |\varphi_j\rangle\langle\varphi_j| = \delta_{ij}.$$

It follows that the support of the state $\Phi'(|\varphi_i\rangle\langle\varphi_i|)$ belongs to the subspace $\{\lambda|\varphi_i\rangle\} \otimes \mathcal{H}_B$ and hence $\text{Tr}_{\mathcal{H}_B} \Phi'(|\varphi_i\rangle\langle\varphi_i|) = |\varphi_i\rangle\langle\varphi_i|$ for all i . \square

4 Arbitrary families of pure states

In this section we consider a structure of a quantum channel reversible with respect to an arbitrary complete family $\mathfrak{S} = \{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ of pure states.

It is known that a channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is reversible with respect to the family of all pure states in $\mathfrak{S}(\mathcal{H}_A)$ (which means that it is reversible with respect to $\mathfrak{S}(\mathcal{H}_A)$) if and only if its complementary channel is completely depolarizing, i.e. if and only if Φ is isometrically equivalent to the channel

$$\Phi'(\rho) = \rho \otimes \sigma \tag{8}$$

from $\mathfrak{S}(\mathcal{H}_A)$ into $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{K})$, where \mathcal{K} is a Hilbert space and σ is a given state in $\mathfrak{S}(\mathcal{K})$ [4, 7].

We give first a characterization of a family $\mathfrak{S} = \{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda} \subset \mathfrak{S}(\mathcal{H}_A)$ such that the reversibility of a channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ with respect to \mathfrak{S} implies its reversibility with respect to $\mathfrak{S}(\mathcal{H}_A)$.

Definition 5. A family $\{|\varphi_\lambda\rangle\}_{\lambda \in \Lambda}$ of vectors in \mathcal{H} (corresp. a family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ of pure states in $\mathfrak{S}(\mathcal{H})$) is called *orthogonally decomposable* if

these is a proper subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that the vector $|\varphi_\lambda\rangle$ lies either in \mathcal{H}_0 or in \mathcal{H}_0^\perp for each $\lambda \in \Lambda$.

Families of pure states, which are not orthogonally decomposable, will be called *orthogonally non-decomposable* (briefly, OND) families.

Proposition 2. *Let $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ be a complete family of pure states in $\mathfrak{S}(\mathcal{H}_A)$. The following statements are equivalent:*

- (i) *the family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ is orthogonally non-decomposable;*
- (ii) *any channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ reversible with respect to the family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ is isometrically equivalent to channel (8).*

Proof. (i) \Rightarrow (ii) If $\Psi : \mathfrak{S}(\mathcal{H}_B) \rightarrow \mathfrak{S}(\mathcal{H}_A)$ is a reverse channel for the channel Φ then Lemma 2 below shows that $\Psi \circ \Phi = \text{Id}_{\mathcal{H}_A}$. Thus the channel Φ is reversible with respect to the set $\mathfrak{S}(\mathcal{H}_A)$ and hence its complementary channel $\hat{\Phi}$ is a completely depolarizing channel.

(ii) \Rightarrow (i) If \mathcal{H}_0 is a proper subspace of \mathcal{H}_A such that the vector $|\varphi_\lambda\rangle$ lies either in \mathcal{H}_0 or in \mathcal{H}_0^\perp for each $\lambda \in \Lambda$ then the channel $\rho \mapsto P_0 \rho P_0 + \bar{P}_0 \rho \bar{P}_0$, where P_0 is the projector on \mathcal{H}_0 and $\bar{P}_0 = I_{\mathcal{H}_A} - P_0$, is obviously reversible with respect to the family $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$. \square

Lemma 2. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \rightarrow \mathfrak{S}(\mathcal{H})$ be a quantum channel ($\dim \mathcal{H} \leq +\infty$) and $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ be an orthogonally non-decomposable family of pure states in $\mathfrak{S}(\mathcal{H})$. If $\Phi(|\varphi_\lambda\rangle\langle\varphi_\lambda|) = |\varphi_\lambda\rangle\langle\varphi_\lambda|$ for all $\lambda \in \Lambda$ then $\Phi|_{\mathfrak{S}(\mathcal{H}_0)} = \text{Id}_{\mathcal{H}_0}$, where \mathcal{H}_0 is the subspace generated by the family $\{|\varphi_\lambda\rangle\}_{\lambda \in \Lambda}$.*

Proof. Let $\Phi(\rho) = \text{Tr}_{\mathcal{K}} V \rho V^*$ be the Stinespring representation of the channel Φ , where V is an isometry from \mathcal{H} into $\mathcal{H} \otimes \mathcal{K}$.

By using the standard argumentation based on Zorn's lemma one can show that any complete OND family of pure states contains a countable complete OND subfamily (Lemma 4 in the Appendix).

Let $\{|\varphi_i\rangle\langle\varphi_i|\}$ be a countable OND subfamily of $\{|\varphi_\lambda\rangle\langle\varphi_\lambda|\}_{\lambda \in \Lambda}$ such that the family $\{|\varphi_i\rangle\}$ generates the subspace \mathcal{H}_0 . The condition of the lemma implies

$$V|\varphi_i\rangle = |\varphi_i\rangle \otimes |\psi_i\rangle, \quad \forall i,$$

where $\{|\psi_i\rangle\}$ is a family of unit vectors in \mathcal{K} . Since V is an isometry, we have

$$\langle\varphi_i|\varphi_j\rangle = \langle V\varphi_i|V\varphi_j\rangle = \langle\varphi_i|\varphi_j\rangle\langle\psi_i|\psi_j\rangle, \quad \forall i, j$$

and hence $\langle\varphi_i|\varphi_j\rangle \neq 0 \Rightarrow \langle\psi_i|\psi_j\rangle = 1$.

It follows that $|\psi_i\rangle = |\psi_j\rangle$ for all i, j . Indeed, if there exist index sets I and J such that $|\psi_i\rangle \neq |\psi_j\rangle$ for all $i \in I, j \in J$ then the above implication shows that $\langle \varphi_i | \varphi_j \rangle = 0$ for all $i \in I, j \in J$ contradicting to the assumed orthogonal non-decomposability of the family $\{|\varphi_i\rangle\langle\varphi_i|\}$.

Thus we have $V|\varphi_i\rangle = |\varphi_i\rangle \otimes |\psi\rangle$ for all i and hence $V|\varphi\rangle = |\varphi\rangle \otimes |\psi\rangle$ for all $|\varphi\rangle \in \mathcal{H}_0$, since the family $\{|\varphi_i\rangle\}$ generates the subspace \mathcal{H}_0 . It follows that $\Phi(\rho) = \rho$ for all $\rho \in \mathfrak{S}(\mathcal{H}_0)$. \square

In analysis of reversibility of a channel with respect to orthogonally decomposable families of pure states the following simple observation plays an essential role.

Lemma 3. *An arbitrary family \mathfrak{S} of pure states in $\mathfrak{S}(\mathcal{H})$ can be decomposed as follows $\mathfrak{S} = \bigcup_k \mathfrak{S}_k$, where $\{\mathfrak{S}_k\}$ is a finite or countable collection of OND disjoint subfamilies of \mathfrak{S} such that $\rho \perp \rho'$ for all $\rho \in \mathfrak{S}_k, \rho' \in \mathfrak{S}_{k'}, k \neq k'$. This decomposition is unique (up to permutation of the subfamilies).*

Proof. For given $\rho \in \mathfrak{S}$ consider the monotone sequence $\{\mathfrak{C}_n^\rho\}$ of subfamilies of \mathfrak{S} constructed as follows. Let $\mathfrak{C}_1^\rho = \{\rho\}$, \mathfrak{C}_2 be the family of all states from \mathfrak{S} non-orthogonal to ρ , \mathfrak{C}_{n+1} be the family of all states from \mathfrak{S} non-orthogonal to at least one state from \mathfrak{C}_n , $n = 2, 3, \dots$. Let $\mathfrak{C}_*^\rho = \bigcup_n \mathfrak{C}_n^\rho$. It is easy to verify by induction that \mathfrak{C}_n^ρ is an OND family for each n and hence \mathfrak{C}_*^ρ is an OND family. Note that any state in \mathfrak{C}_*^ρ is orthogonal to any state in $\mathfrak{S} \setminus \mathfrak{C}_*^\rho$. Indeed, if $\rho \in \mathfrak{C}_*^\rho$ then $\rho \in \mathfrak{C}_n^\rho$ for some n . So, if a pure state σ is not orthogonal to ρ then it belongs to $\mathfrak{C}_{n+1}^\rho \subseteq \mathfrak{C}_*^\rho$.

It is easy to see that the families \mathfrak{C}_*^ρ and $\mathfrak{C}_*^{\rho'}$, $\rho, \rho' \in \mathfrak{S}$, either coincide or have an empty intersection. Since the Hilbert space \mathcal{H} is separable and each family \mathfrak{C}_*^ρ occupies a nontrivial subspace of \mathcal{H} , the collection $\{\mathfrak{C}_*^\rho\}_{\rho \in \mathfrak{S}}$ contains either a finite or countable number of different families. These families form the required decomposition. \square

The above decomposition of a complete family \mathfrak{S} of pure states provides a description of the class of all channels reversible with respect to \mathfrak{S} .

Theorem 4. *Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel and \mathfrak{S} be a complete family of pure states in $\mathfrak{S}(\mathcal{H}_A)$. Let $\mathfrak{S} = \bigcup_k \mathfrak{S}_k$ be a decomposition of \mathfrak{S} into OND subfamilies (from Lemma 3) and P_k – the projector on the subspace generated by the states in \mathfrak{S}_k . The following statements are equivalent:*

- (i) *the channel Φ is reversible with respect to the family \mathfrak{S} ;*

(ii) the channel Φ is reversible with respect to the family

$$\hat{\mathfrak{S}} = \left\{ \rho \in \mathfrak{S}(\mathcal{H}_A) \mid \rho = \sum_k P_k \rho P_k \right\};$$

(iii) $\hat{\Phi}$ is a c - q channel having the representation $\hat{\Phi}(\rho) = \sum_k [\text{Tr} P_k \rho] \sigma_k$,
where $\{\sigma_k\}$ is a set of states in $\mathfrak{S}(\mathcal{H}_E)$ such that $\text{rank} \sigma_k \leq \dim \mathcal{H}_B \forall k$;

(iv) the channel Φ is isometrically equivalent to the channel

$$\Phi'(\rho) = \sum_{k,l} P_k \rho P_l \otimes \sum_{p,t} \langle \psi_t^l | \psi_p^k \rangle |p\rangle \langle t|$$

from $\mathfrak{S}(\mathcal{H}_A)$ into $\mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\{|\psi_p^k\rangle\}$ is a collection of vectors in a separable Hilbert space such that $\sum_p \|\psi_p^k\|^2 = 1 \forall k$ and $\{|p\rangle\}$ is an orthonormal basis in \mathcal{H}_B .

Proof. (i) \Rightarrow (ii). Let Ψ be a channel such that $\Psi(\Phi(\rho)) = \rho$ for all $\rho \in \mathfrak{S}$. Let \mathcal{H}_k be the subspace of \mathcal{H} generated by the vectors corresponding to the subfamily \mathfrak{S}_k . Since \mathfrak{S}_k is an OND family, Lemma 2 shows that $\Psi \circ \Phi|_{\mathfrak{S}(\mathcal{H}_k)} = \text{Id}_{\mathcal{H}_k}$ for each k .

(ii) \Rightarrow (iii). Let $\{|\phi_i\rangle\}$ be an orthonormal basis corresponding to the decomposition $\mathcal{H}_A = \oplus_k \mathcal{H}_k$, i.e. each $|\phi_i\rangle$ lies in some \mathcal{H}_k . Let I_k be the set of all i such that $|\phi_i\rangle \in \mathcal{H}_k$. Since $|\phi_i\rangle \langle \phi_i| \in \hat{\mathfrak{S}}$ for all i , the channel Φ is reversible with respect to the family $\{|\phi_i\rangle \langle \phi_i|\}$. By Proposition 1 we have

$$\hat{\Phi}(\rho) = \sum_k \sum_{i \in I_k} \langle \phi_i | \rho | \phi_i \rangle \sigma_i,$$

where $\{\sigma_i\}$ is a set of states in $\mathfrak{S}(\mathcal{H}_E)$ such that $\text{rank} \sigma_i \leq \dim \mathcal{H}_B$ for all i . Since \mathfrak{S}_k is an OND family, Proposition 2 shows that the restriction of the channel $\hat{\Phi}$ to the set $\mathfrak{S}(\mathcal{H}_k)$ is a completely depolarizing channel. Hence $\sigma_i = \bar{\sigma}_k$ for all $i \in I_k$. Thus $\hat{\Phi}(\rho) = \sum_k \text{Tr}[P_k \rho] \bar{\sigma}_k$.

(iii) \Rightarrow (iv). Let $k(i)$ be the index of the set I_k containing i , i.e. $i \in I_{k(i)}$ for all i . If $\sigma_k = \sum_{p=1}^{\dim \mathcal{H}_B} |\psi_p^k\rangle \langle \psi_p^k|$ then $\hat{\Phi}(\rho) = \sum_{i,p} W_{ip} \rho W_{ip}^*$, where $W_{ip} = |\psi_p^{k(i)}\rangle \langle \phi_i|$, and hence representation (4) implies

$$\begin{aligned} \hat{\Phi}(\rho) &= \sum_{i,j,p,t} [\text{Tr} W_{ip} \rho W_{jt}^*] |\phi_i\rangle \langle \phi_j| \otimes |p\rangle \langle t| = \\ &= \sum_{k,l,p,t} \sum_{i \in I_k, j \in I_l} \langle \phi_i | \rho | \phi_j \rangle |\phi_i\rangle \langle \phi_j| \otimes \langle \psi_t^l | \psi_p^k \rangle |p\rangle \langle t| = \sum_{k,l} P_k \rho P_l \otimes \sum_{p,t} \langle \psi_t^l | \psi_p^k \rangle |p\rangle \langle t|, \end{aligned}$$

where $\{|p\rangle\}$ is an orthonormal basis in \mathcal{H}_B .

(iv) \Rightarrow (i) follows from Lemma 1, since $\Psi(\cdot) = \text{Tr}_{\mathcal{H}_B}(\cdot)$ is the "reverse" channel for the channel Φ' with respect to the family \mathfrak{S} . \square

Theorem 4 implies the following useful observation.

Corollary 2. *If a channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is reversible with respect to a complete family \mathfrak{S} of pure states in $\mathfrak{S}(\mathcal{H}_A)$ then it is reversible with respect to a particular complete family of orthogonal pure states in $\mathfrak{S}(\mathcal{H}_A)$.*

Remark 1. If the complete family of pure states \mathfrak{S} contains a subfamily $\mathfrak{S}_0 = \{|\varphi_i\rangle\langle\varphi_i|\}$ such that $\{|\varphi_i\rangle\}$ is a basis in the space \mathcal{H}_A (in the sense that an arbitrary vector $|\psi\rangle$ has a unique decomposition $|\psi\rangle = \sum_i c_i |\varphi_i\rangle$)⁴ then the family of orthogonal pure states mentioned in Corollary 2 is explicitly given by Theorem 3. Indeed, by Lemma 5 in the Appendix the set $\{|\phi_i\rangle\}$ of vectors defined in (5) by means of an arbitrary non-degenerate probability distribution $\{\pi_i\}$ forms an orthonormal basis in \mathcal{H}_A . It is easy to see that the channel Φ' defined in (6) is reversible with respect to the family $\{|\phi_i\rangle\langle\phi_i|\}$. By Theorem 3 and Lemma 1 the same property holds for the channel Φ .

In the case of a finite dimensional channel with the same input and output ($\dim \mathcal{H}_A = \dim \mathcal{H}_B < +\infty$) Theorem 4 implies the following observation.

Corollary 3. *Let $\Phi : \mathfrak{S}(\mathcal{H}) \rightarrow \mathfrak{S}(\mathcal{H})$ be a quantum channel, where $n = \dim \mathcal{H} < +\infty$, and \mathfrak{S} be a complete family of pure states in $\mathfrak{S}(\mathcal{H})$. Let $\mathfrak{S} = \bigcup_k \mathfrak{S}_k$ be a decomposition of \mathfrak{S} into OND subfamilies and P_k – the projector on the subspace generated by the states in \mathfrak{S}_k . The channel Φ is reversible with respect to the family \mathfrak{S} if and only if it is unitary equivalent to the channel*

$$\Phi'(\rho) = \sum_{k,l} c_{kl} P_k \rho P_l, \quad \rho \in \mathfrak{S}(\mathcal{H}),$$

where $\|c_{kl}\|$ is a Gram matrix of a collection of unit vectors.

Proof. Since $\Phi'(\rho) = \rho$ for all $\rho \in \mathfrak{S}$, it suffices to prove the "only if" part of the corollary.

By Corollary 2 the reversibility of the channel Φ with respect to \mathfrak{S} implies its reversibility with respect to some family $\{\rho_i\}_{i=1}^n$ of orthogonal pure states

⁴Existence of the subfamily \mathfrak{S}_0 is obvious if and only if \mathcal{H}_A is a finite dimensional space. The condition showing that a complete countable family of unit vectors in an infinite dimensional Hilbert space forms a basis can be found in [1, Chapter I].

in $\mathfrak{S}(\mathcal{H})$. Hence

$$\frac{1}{n} \sum_{i=1}^n H(\Phi(\rho_i) \| \Phi(\bar{\rho})) = \frac{1}{n} \sum_{i=1}^n H(\rho_i \| \bar{\rho}) = \log n,$$

where $\bar{\rho} = n^{-1}I_{\mathcal{H}}$ and $H(\cdot \| \cdot)$ is the relative entropy. It follows that the family $\{\Phi(\rho_i)\}_{i=1}^n$ consists of orthogonal pure states and that $\Phi(I_{\mathcal{H}}) = I_{\mathcal{H}}$.

Hence, by definition of the complementary channel, $\{\widehat{\Phi}(\rho_i)\}_{i=1}^n$ is a family of pure states and Theorem 4 shows that $\widehat{\Phi}(\rho) = \sum_k [\text{Tr } P_k \rho] |\psi_k\rangle\langle\psi_k|$, where $\{|\psi_k\rangle\}$ is a set of unit vectors in \mathcal{H}_E . It follows that the channel Φ is isometrically equivalent to the channel $\widehat{\widehat{\Phi}} = \Phi'$ with $c_{kl} = \langle\psi_l | \psi_k\rangle$. Since the both channels are unital, their isometrical equivalence means unitary equivalence. \square

Remark 2. Corollary 2 shows that in the case $\dim \mathcal{H}_A = \dim \mathcal{H}_B < +\infty$ a channel Φ is reversible with respect to a complete family \mathfrak{S} of pure states if and only if $\Phi(\rho) = U\rho U^*$ for all $\rho \in \mathfrak{S}$, where U is an unitary operator, i.e. the reversibility of a channel with respect to a complete family of pure states is equivalent to *preserving* of all states of the family by this channel (up to unitary transformation).

5 Applications

By using the above observations one can strengthen the results concerning several entropic and informational characteristics of a quantum channel presented in [13, Section 5].

These results are based on the following corollary of the Petz's theorem: *preserving of the Holevo quantity $\chi(\mu)$ of a generalized ensemble μ of quantum states in $\mathfrak{S}(\mathcal{H}_A)$ (defined as a Borel probability measure on the set $\mathfrak{S}(\mathcal{H}_A)$) under action of a quantum channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ is equivalent to reversibility of this channel with respect to μ -almost all ρ in $\mathfrak{S}(\mathcal{H}_A)$ (see details in Section 4 in [13]).*

This equivalence and the results of Sections 3 and 4 imply the following criterion for preserving of the Holevo quantity (strengthening the second assertion of Theorem 2 in [13]).

Theorem 5. *Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel. The following statements are equivalent:*

- (i) *there exists a generalized ensemble μ of pure states in $\mathfrak{S}(\mathcal{H}_A)$ with the full rank average state $\bar{\rho}(\mu)$ such that $\chi(\Phi(\mu)) = \chi(\mu) < +\infty$;*
- (ii) *there exists a complete family $\{|\varphi_i\rangle\langle\varphi_i|\}$ of orthogonal pure states in $\mathfrak{S}(\mathcal{H}_A)$ such that equivalent statements (i) – (iii) of Proposition 1 hold.*

This theorem provides the following specifications of Proposition 2 and of Corollary 2 in [13] concerning the properties of the Holevo capacity $\bar{C}(\Phi)$ and of the minimal output entropy $H_{\min}(\Phi)$ of a quantum channel Φ .

Corollary 4. *Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel such that $\dim \mathcal{H}_A < +\infty$ and $\hat{\Phi} : \mathfrak{S}(\mathcal{H}_A) \rightarrow \mathfrak{S}(\mathcal{H}_E)$ be its complementary channel.*

A) *The equality in the general inequality*

$$\bar{C}(\Phi) \leq \log \dim \mathcal{H}_A \quad (9)$$

holds if and only if $\hat{\Phi}$ is a c-q channel, which means that the channel Φ is isometrically equivalent to the pseudo-diagonal channel described in statement (iii) of Proposition 1 with a particular orthonormal basis $\{|\varphi_i\rangle\}$ in \mathcal{H}_A .

B) *If $\mathcal{H}_B = \mathcal{H}_A$ then the equality in (9) holds if and only if the channel Φ is unitary equivalent to the channel Φ' described in Corollary 3.*

C) *If $\mathcal{H}_B = \mathcal{H}_A$ and the channel Φ is covariant with respect to some irreducible representation $\{V_g\}_{g \in G}$ of a compact group G in the sense that $\Phi(V_g \rho V_g^*) = V_g \Phi(\rho) V_g^*$ for all $g \in G$ then $H_{\min}(\Phi) = 0$ if and only if the channel Φ is unitary equivalent to the channel Φ' described in Corollary 3.*

The condition of assertion C of Corollary 4 holds for any unital qubit channel Φ .

By using the above Theorem 5 instead of Theorem 2 in [13] one can specify the necessary condition for coincidence of the constrained Holevo capacity $\bar{C}(\Phi, \rho)$ and the quantum mutual information $I(\Phi, \rho)$ of a quantum channel Φ at a state ρ presented in Proposition 4 in [13].

Corollary 5. *If the conditions of Proposition 4 in [13] hold then*

$$\bar{C}(\Phi, \rho) = I(\Phi, \rho) < +\infty \quad \Rightarrow \quad \Phi|_{\mathfrak{S}(\mathcal{H}_\rho)} \text{ is a c-q channel} \quad (\mathcal{H}_\rho = \text{supp} \rho).$$

By using this corollary one can specify the necessary condition for coincidence of the Holevo capacity and the entanglement assisted classical capacity of a (constrained or unconstrained) quantum channel presented in [12, Proposition 2] and [13, Corollary 4]. This specification consists in replacing the term "entanglement-breaking" by the term "classical-quantum".

Appendix

Lemma 4. *An arbitrary complete orthogonally non-decomposable family of pure states in a separable Hilbert space \mathcal{H} contains a countable complete orthogonally non-decomposable subfamily.*

Proof. Let \mathfrak{H} be the set of all subspaces of \mathcal{H} generated by countable OND subfamilies of the family \mathfrak{S} endowed with the inclusion ordering. Let \mathfrak{H}_0 be a chain in \mathfrak{H} and $\mathcal{H}_0 = \overline{\bigcup_{\mathcal{K} \in \mathfrak{H}_0} \mathcal{K}}$. Since there is a countable chain $\{\mathcal{H}_k\}$ in \mathfrak{H} such that $\mathcal{H}_0 = \overline{\bigcup_k \mathcal{H}_k}$ and a countable union of countable OND subfamilies is a countable OND subfamily, the subspace \mathcal{H}_0 belongs to the set \mathfrak{H} . Hence \mathcal{H}_0 is an upper bound of the chain \mathfrak{H}_0 and Zorn's lemma implies existence of a maximal element \mathcal{H}_m in \mathfrak{H} . Suppose, $\mathcal{H}_m \subsetneq \mathcal{H}$. Since the family \mathfrak{S} is complete and orthogonally non-decomposable, it contains a pure state $|\varphi\rangle\langle\varphi|$ such that the vector $|\varphi\rangle$ lies neither in \mathcal{H}_m nor in \mathcal{H}_m^\perp . By adding the state $|\varphi\rangle\langle\varphi|$ to the countable OND subfamily corresponding to the subspace \mathcal{H}_m we obtain a countable OND subfamily. Hence $\mathcal{H}_m \vee \{\lambda|\varphi\rangle\} \in \mathfrak{H}$ contradicting to the maximality of \mathcal{H}_m \square .

Lemma 5. *Let $\{|\varphi_i\rangle\}$ be a basis in a Hilbert space \mathcal{H} (in the sense that an arbitrary vector $|\psi\rangle$ in \mathcal{H} has a unique decomposition $|\psi\rangle = \sum_i c_i |\varphi_i\rangle$). Then the set $\{|\phi_i\rangle\}$ of vectors defined in (5) by means of an arbitrary non-degenerate probability distribution $\{\pi_i\}$ is an orthonormal basis in \mathcal{H} .*

Proof. Since $\sum_i |\phi_i\rangle\langle\phi_i| = I_{\mathcal{H}}$, for given arbitrary j we have

$$|\phi_j\rangle = \sum_i \langle\phi_i|\phi_j\rangle |\phi_i\rangle$$

and hence

$$(\|\phi_j\|^2 - 1)|\phi_j\rangle + \sum_{i \neq j} \langle\phi_i|\phi_j\rangle |\phi_i\rangle = 0.$$

By multiplying the both sides of this vector equality by the operator $\bar{\rho}_\pi$

defined in (5) we obtain

$$\sqrt{\pi_j}(\|\phi_j\|^2 - 1)|\varphi_j\rangle + \sum_{i \neq j} \sqrt{\pi_i} \langle \phi_i | \phi_j \rangle |\varphi_i\rangle = 0.$$

Since $\{|\varphi_i\rangle\}$ is a basis and $\pi_i > 0$ for all i , we have $\|\phi_j\|^2 = 1$ and $\langle \phi_i | \phi_j \rangle = 0$ for all $i \neq j$. Thus $\{|\phi_i\rangle\}$ is an orthonormal system of vectors in \mathcal{H} . It is a complete system, since $\sum_i |\phi_i\rangle \langle \phi_i| = I_{\mathcal{H}}$. \square

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